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Knudsen Layer for Gas Mixtures

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The Knudsen layer in rarefied gas dynamics is essentially described by a half-space boundary-value problem of the linearized Boltzmann equation, in which the incoming data are specified on the boundary and the solution is assumed to be bounded at infinity (Milne problem). This problem is considered for a binary mixture of hard-sphere gases, and the existence and uniqueness of the solution, as well as some asymptotic properties, are proved. The proof is an extension of that of the corresponding theorem for a single-component gas given by Bardos, Caflisch, and Nicolaenko [*Comm. Pure Appl. Math.* **39**:323 (1986)]. Some estimates on the convergence of the solution in a finite slab to the solution of the Milne problem are also obtained.

KEY WORDS: Knudsen layer; gas mixtures; Milne problem; Boltzmann equation; rarefied gas dynamics.

1. INTRODUCTION

The behavior of a rarefied gas around bodies is described by the Boltzmann equation. The steady behavior of the gas when the Knudsen number, the ratio of the mean free path of the gas molecules to the characteristic length of the system, is small has thoroughly been investigated by Sone by means of a systematic asymptotic analysis.[10–13] According to this asymptotic theory, the solution of the Boltzmann equation and its kinetic boundary condition is expressed as a sum of two parts: one is the moderately varying

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overall solution whose macroscopic variables are described by the fluid-dynamic-type equations, and the other is the correction term that is appreciable only in a thin layer with thickness of a few mean free paths adjacent to the boundary (Knudsen-layer). We call the former the hydrodynamic solution and the latter the Knudsen-layer correction. The appropriate boundary conditions for the fluid-dynamic-type equations are determined by the analysis of the Knudsen-layer correction (see the following paragraph). The types of the equations and boundary conditions differ depending on the physical situation under consideration. However, except for the case where strong evaporation or condensation is taking place on the boundary, the analysis of the Knudsen layer is essentially reduced to that of the half-space boundary-value problem of the linearized Boltzmann equation with or without a source term. The reader is referred to ref. 13 for comprehensive and detailed description of the asymptotic theory.

The half-space problem for the Knudsen-layer correction is such that the boundary data for the incoming molecules contain the values of the macroscopic variables of the hydrodynamic solution on the boundary. These boundary values are determined together with the Knudsen-layer correction which vanishes rapidly outside the Knudsen layer. The boundary values thus obtained give the appropriate boundary conditions for the fluid-dynamic-type equations. In other words, the matching of the Knudsen layer with the domain outside the layer is made by selecting the boundary conditions for the fluid-dynamic-type equations that make the Knudsen-layer correction vanish outside the Knudsen layer.

The existence and uniqueness of the solution for the Knudsen-layer correction as well as the boundary values of the hydrodynamic solution was conjectured by Grad[6] in 1969 (see also ref. 13). However, his conjecture was proved much later, in a slightly different but equivalent formulation, by Bardos, Caflisch, and Nicolaenko,[1] Cercignani,[2] Coron, Golse, and Sulem,[4] Golse and Poupaud,[5] and Maslova.[7] For instance, Bardos, Caflisch, and Nicolaenko[1] considered the Milne and Kramers problems for the linearized Boltzmann equation for hard-sphere molecules with or without a source term, that is, the half-space boundary-value problem in which the velocity distribution for the incoming molecules from the boundary is specified and that at infinity is assumed to be bounded (Milne problem) or linearly growing (Kramers problem). (The names Milne and Kramers problems are used in different ways depending on the authors. Here, we follow Bardos, Caflisch, and Nicolaenko[1]). In the mean time, extensive numerical analysis of these problems has been carried out in many papers in connection with the physically fundamental problems, such as the temperature jump, shear slip, and thermal creep (see, for example, refs. 8 and 14).

The works that we mentioned above are all for a single-component gas. Recently, Takata and Aoki investigated, by a systematic asymptotic analysis of the Boltzmann equation, the steady behavior of binary gas mixtures for small Knudsen numbers.(17) They considered the mixtures around solid bodies or condensed phases of one of the component gases in the situation where the Mach number of the flow is as small as the Knudsen number while the variation of the other macroscopic variables may be large, and derived a set of fluid-dynamic-type equations and boundary conditions. In the derivation of the boundary conditions, they presented and used the conjectured theorem, corresponding to Grad's conjecture for a single-component gas, for the Knudsen-layer correction for a gas mixture (see Appendix B of ref. 17).

The aim of the present paper is to give a proof of the conjectured theorem for a binary mixture of hard-sphere gases. The paper is organized as follows. After summarizing some basic properties of the Boltzmann equation in Section 2, we first prove, in Section 3, the existence and uniqueness theorem for the Milne problem (without a source term) and some asymptotic properties of the solution. The proof follows that of Bardos, Caflisch, and Nicolaenko(1) for a single-component gas. In Section 4, we consider the Kramers problem and show that it can be reduced to the Milne problem. The Milne problem with a source term is dealt with in Section 5. Finally in Section 6, we give some estimates on the convergence of the solution in a finite slab to the solution of the Milne problem.

2. DESCRIPTION

Consider a mixture of two types of hard-sphere gases: gas A and gas B. The indexes $\alpha \in \{A, B\}$ and $\beta \in \{A, B\}$ are systematically used. The m^α and d^α are the (dimensionless) mass and diameter of a molecule of component gas α.

The Boltzmann equation for such a binary mixture is

$$\partial_t F^A + \zeta \nabla_x F^A = J^{AA}(F^A, F^A) + J^{BA}(F^B, F^A), \tag{1}$$

$$\partial_t F^B + \zeta \nabla_x F^B = J^{AB}(F^A, F^B) + J^{BB}(F^B, F^B), \tag{2}$$

where the bilinear operators $J^{\alpha\beta}$ represent the elastic collisions. More precisely for any $\alpha, \beta \in \{A, B\}$ one has

$$J^{\beta\alpha}(F^\beta, F^\alpha)(\zeta) = \iint [F^\beta(\zeta'_*)\, F^\alpha(\zeta') - F^\beta(\zeta_*)\, F^\alpha(\zeta)]\, B^{\beta\alpha}\, d\omega\, d\zeta_* \tag{3}$$

with

$$B^{\beta\alpha} = \frac{1}{4\sqrt{2\pi}} \left(\frac{d^\alpha + d^\beta}{2}\right)^2 |(\zeta_* - \zeta)\cdot\omega|, \qquad \omega \in S^2, \tag{4}$$

$$\zeta' = \zeta'^{(\beta\alpha)} = \zeta + \frac{2m^\beta}{m^\alpha + m^\beta} [(\zeta_* - \zeta)\cdot\omega]\,\omega, \tag{5}$$

$$\zeta'_* = \zeta'^{(\beta\alpha)}_* = \zeta_* - \frac{2m^\alpha}{m^\alpha + m^\beta} [(\zeta_* - \zeta)\cdot\omega]\,\omega. \tag{6}$$

Observe that the mapping (associated to the operator $J^{\beta\alpha}$) $(\zeta, \zeta_*) \mapsto (\zeta', \zeta'_*)$ is an involution which preserves (the mass,) the momentum and the energy:

$$m^\alpha\zeta' + m^\beta\zeta'_* = m^\alpha\zeta + m^\beta\zeta_*, \tag{7}$$

$$\tfrac{1}{2}m^\alpha |\zeta'|^2 + \tfrac{1}{2}m^\beta |\zeta'_*|^2 = \tfrac{1}{2}m^\alpha |\zeta|^2 + \tfrac{1}{2}m^\beta |\zeta_*|^2. \tag{8}$$

For $F = (F^A, F^B)$ the collision operator $\mathscr{C}$ is therefore given by

$$\mathscr{C}(F) = \begin{pmatrix} J^{AA}(F^A, F^A) + J^{BA}(F^B, F^A) \\ J^{AB}(F^A, F^B) + J^{BB}(F^B, F^B) \end{pmatrix} \tag{9}$$

and the "two species" Boltzmann equation can be written in a more synthetic form according to the formula:

$$\partial_t F + \zeta \nabla_x F = \mathscr{C}(F). \tag{10}$$

The classical invariance property of the Boltzmann collision operator is generalized as follows:

Proposition 2.1. For any pair of smooth functions $G = (G^A, G^B)$ one has

$$\begin{aligned}(\mathscr{C}(F), G)_{(L^2(\mathbb{R}^3))^2} = &-\tfrac{1}{4} I_{AA}(F^A, G^A) - \tfrac{1}{2} I_{AB}(F^A, F^B, G^A, G^B) \\ &-\tfrac{1}{4} I_{BB}(F^B, G^B)\end{aligned} \tag{11}$$

with

$$\begin{aligned}I_{AA}(F^A, G^A) = \iiint &[F^A(\zeta'_*)\,F^A(\zeta') - F^A(\zeta_*)\,F^A(\zeta)] \\ &\times [G^A(\zeta'_*) + G^A(\zeta') - G^A(\zeta_*) - G^A(\zeta)]\,B^{AA}\,d\omega\,d\zeta_*\,d\zeta,\end{aligned} \tag{12}$$

$$I_{AB}(F^A, F^B, G^A, G^B) = \iiint [F^A(\zeta'_*)\, F^B(\zeta') - F^A(\zeta_*)\, F^B(\zeta)] \times [G^A(\zeta'_*) + G^B(\zeta') - G^A(\zeta_*) - G^B(\zeta)]\, B^{AB}\, d\omega\, d\zeta_*\, d\zeta, \tag{13}$$

$$I_{BB}(F^B, G^B) = \iiint [F^B(\zeta'_*)\, F^B(\zeta') - F^B(\zeta_*)\, F^B(\zeta)] \times [G^B(\zeta'_*) + G^B(\zeta') - G^B(\zeta_*) - G^B(\zeta)]\, B^{BB}\, d\omega\, d\zeta_*\, d\zeta. \tag{14}$$

The proof left to the reader is similar to the classical one in the case of one species.

Corollary 2.1. For any function F one has

$$(\mathscr{C}(F), \log F)_{(L^2(\mathbb{R}^3))^2} = \left(\mathscr{C}(F), \begin{pmatrix} \log F^A \\ \log F^B \end{pmatrix}\right)_{(L^2(\mathbb{R}^3))^2} \leqslant 0 \tag{15}$$

and the equality holds if and only if F is "bi-Maxwellian:"

$$F = \begin{pmatrix} \dfrac{\rho^A}{(2\pi T)^{\frac{3}{2}}}\, e^{-m^A \frac{|\zeta - u|^2}{2T}} \\ \dfrac{\rho^B}{(2\pi T)^{\frac{3}{2}}}\, e^{-m^B \frac{|\zeta - u|^2}{2T}} \end{pmatrix}. \tag{16}$$

Proof. The Proposition 2.1 is used with $G^A = \log F^A$ and $G^B = \log F^B$, and one has

$$I_{AA}(F^A, \log F^A) = \iiint [F^A(\zeta'_*)\, F^A(\zeta') - F^A(\zeta_*)\, F^A(\zeta)] \times \log\left[\frac{F^A(\zeta'_*)\, F^A(\zeta')}{F^A(\zeta_*)\, F^A(\zeta)}\right] B^{AA}\, d\omega\, d\zeta_*\, d\zeta, \tag{17}$$

$$I_{AB}(F^A, F^B, \log F^A, \log F^B) = \iiint [F^A(\zeta'_*)\, F^B(\zeta') - F^A(\zeta_*)\, F^B(\zeta)] \times \log\left[\frac{F^A(\zeta'_*)\, F^B(\zeta')}{F^A(\zeta_*)\, F^B(\zeta)}\right] B^{AB}\, d\omega\, d\zeta_*\, d\zeta, \tag{18}$$

$$I_{BB}(F^B, \log F^B) = \iiint [F^B(\zeta'_*)\, F^B(\zeta') - F^B(\zeta_*)\, F^B(\zeta)] \times \log\left[\frac{F^B(\zeta'_*)\, F^B(\zeta')}{F^B(\zeta_*)\, F^B(\zeta)}\right] B^{BB}\, d\omega\, d\zeta_*\, d\zeta. \tag{19}$$

Each of the above three terms is non-negative and therefore

$$(\mathscr{C}(F), \log F)_{(L^2(\mathbb{R}^3))^2} \leqslant 0. \tag{20}$$

Now if in (20) the equality holds one has

$$0 = I_{AA}(F^A, \log F^A) = I_{BB}(F^B, \log F^B) = I_{AB}(F^A, F^B, \log F^A, \log F^B). \tag{21}$$

The first two relations of (21) imply that F^A and F^B are Maxwellians:

$$F^A = \frac{\rho^A}{(2\pi T^A)^{\frac{3}{2}}} e^{-\frac{|\zeta - u^A|^2}{2T^A}}, \qquad F^B = \frac{\rho^B}{(2\pi T^B)^{\frac{3}{2}}} e^{-\frac{|\zeta - u^B|^2}{2T^B}}. \tag{22}$$

With the last relation of (21), one has

$$\begin{aligned} 0 &= \log\left[\frac{F^A(\zeta'_*)\, F^B(\zeta')}{F^A(\zeta_*)\, F^B(\zeta)}\right] \\ &= \log F^A(\zeta'_*) + \log F^B(\zeta') - \log F^A(\zeta_*) - \log F^B(\zeta) \end{aligned} \tag{23}$$

or eventually

$$\frac{|\zeta'_* - u^A|^2}{2T^A} + \frac{|\zeta' - u^B|^2}{2T^B} = \frac{|\zeta_* - u^A|^2}{2T^A} + \frac{|\zeta - u^B|^2}{2T^B} \tag{24}$$

for all (ζ, ζ_*, ζ', and ζ'_*) which satisfy the relations (5) and (6) with $\alpha = B$ and $\beta = A$. This gives, as a consequence of the conservation of momentum and energy,

$$u^A = u^B \qquad \text{and} \qquad m^A T^A = m^B T^B \tag{25}$$

and the proof is completed. ▮

The analysis of the Knudsen layer involves fluctuations near equilibrium states. For the problems of evaporation and condensation one assumes that the bulk velocity is zero and by a change of scale the bulk temperature is taken equal to one. For Maxwellian and bi-Maxwellian the following notations are introduced:

$$M^A = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-m^A \frac{|\zeta|^2}{2}}, \qquad M^B = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-m^B \frac{|\zeta|^2}{2}}, \qquad M = \begin{pmatrix} \rho^A M^A \\ \rho^B M^B \end{pmatrix}. \tag{26}$$

For the linearized version the function $F=(F^A,F^B)$ in (10) is replaced by the function $(\rho^A M^A(1+f^A), \rho^B M^B(1+f^B))$ leading to the linearized version of (10):

$$\partial_t f^A+\zeta\nabla_x f^A=(M^A)^{-1}\,(\rho^A[J^{AA}(M^A,M^Af^A)+J^{AA}(M^Af^A,M^A)] \\ +\rho^B[J^{BA}(M^B,M^Af^A)+J^{BA}(M^Bf^B,M^A)]), \tag{27}$$

$$\partial_t f^B+\zeta\nabla_x f^B=(M^B)^{-1}\,(\rho^A[J^{AB}(M^A,M^Bf^B)+J^{AB}(M^Af^A,M^B)] \\ +\rho^B[J^{BB}(M^B,M^Bf^B)+J^{BB}(M^Bf^B,M^B)]), \tag{28}$$

which is written in a synthetic way according to the formula:

$$\partial_t f+\zeta\nabla_x f+\mathscr{L}(f)=0 \tag{29}$$

with $f\mapsto\mathscr{L}(f)$ being the linear operator defined by the formula:

$$\mathscr{L}(f)=\mathscr{L}\begin{pmatrix} f^A \\ f^B\end{pmatrix} \\ =-\begin{pmatrix} (M^A)^{-1}\,(\rho^A[J^{AA}(M^A,M^Af^A)+J^{AA}(M^Af^A,M^A)] \\ +\rho^B[J^{BA}(M^B,M^Af^A)+J^{BA}(M^Bf^B,M^A)]) \\ (M^B)^{-1}\,(\rho^A[J^{AB}(M^A,M^Bf^B)+J^{AB}(M^Af^A,M^B)] \\ +\rho^B[J^{BB}(M^B,M^Bf^B)+J^{BB}(M^Bf^B,M^B)]) \end{pmatrix}. \tag{30}$$

For the operator $\mathscr{L}$ one introduces the space $\mathscr{H}$ defined by the scalar product:

$$\langle f,g\rangle=\left\langle\begin{pmatrix} f^A \\ f^B\end{pmatrix},\begin{pmatrix} g^A \\ g^B\end{pmatrix}\right\rangle \\ =\int_{\mathbb{R}^3}\rho^A f^A(\zeta)\,g^A(\zeta)\,M^A(\zeta)\,d\zeta+\int_{\mathbb{R}^3}\rho^B f^B(\zeta)\,g^B(\zeta)\,M^B(\zeta)\,d\zeta. \tag{31}$$

The properties of the operator $\mathscr{L}$ which are relevant for the study of the half-space problem are the object of the next.

Proposition 2.2

1. In the space $\mathscr{H}$, $\mathscr{L}$ is the sum of a "diagonal" operator $f\mapsto \nu f$

$$\nu f=\begin{pmatrix} \nu_A(\zeta)\,f^A \\ \nu_B(\zeta)\,f^B\end{pmatrix}\qquad\text{with}\quad \delta(1+|\zeta|)\leqslant\nu_\alpha(|\zeta|)\leqslant\gamma(1+|\zeta|),\quad 0<\delta<\gamma<\infty \tag{32}$$

and of a compact operator

$$\mathscr{K}(f)=\begin{pmatrix}\mathscr{K}^A(f^A,f^B)\\ \mathscr{K}^B(f^B,f^A)\end{pmatrix}=\begin{pmatrix}K^{AA}(f^A)+K^{BA}(f^B)\\ K^{AB}(f^A)+K^{BB}(f^B)\end{pmatrix}. \tag{33}$$

The domain of $\mathscr{L}$ is the space defined by the formula:

$$D(\mathscr{L})=\{f: \|(1+|\zeta|)^{\frac{1}{2}} f\|_{\mathscr{H}}<\infty\} \qquad \text{with} \qquad \|\mathscr{L}f\| \leqslant \gamma_0 \, \|(1+|\zeta|)^{\frac{1}{2}} f\|_{\mathscr{H}}. \tag{34}$$

2. The operator $\mathscr{L}$ is self-adjoint and non-negative, and satisfies the relation

$$\langle \mathscr{L}f, f\rangle=\tfrac{1}{4}\,G_{AA}(f^A,f^A)+\tfrac{1}{2}\,G_{AB}(f^A,f^B)+\tfrac{1}{4}\,G_{BB}(f^B,f^B) \tag{35}$$

with

$$G_{AA}(f^A,f^A)=(\rho^A)^2\iiint[f^A(\zeta'_*)+f^A(\zeta')-f^A(\zeta_*)-f^A(\zeta)]^2 \times M^A(\zeta_*)\,M^A(\zeta)\,B^{AA}\,d\omega\,d\zeta_*\,d\zeta, \tag{36}$$

$$G_{AB}(f^A,f^B)=\rho^A\rho^B\iiint[f^A(\zeta'_*)+f^B(\zeta')-f^A(\zeta_*)-f^B(\zeta)]^2 \times M^A(\zeta_*)\,M^B(\zeta)\,B^{AB}\,d\omega\,d\zeta_*\,d\zeta, \tag{37}$$

and

$$G_{BB}(f^B,f^B)=(\rho^B)^2\iiint[f^B(\zeta'_*)+f^B(\zeta')-f^B(\zeta_*)-f^B(\zeta)]^2 \times M^B(\zeta_*)\,M^B(\zeta)\,B^{BB}\,d\omega\,d\zeta_*\,d\zeta. \tag{38}$$

3. The Kernel of $\mathscr{L}$ (Ker $\mathscr{L}$) is the space of 6 dimension, depending on the parameters

$$n^A,\, n^B,\, u(=(u_1,u_2,u_3)),\, \theta$$

given by the formula

$$\text{Ker}\ \mathscr{L}=\begin{pmatrix}n^A+m^A u\cdot\zeta+\frac{1}{2}\,\theta(m^A\,|\zeta|^2-3)\\ n^B+m^B u\cdot\zeta+\frac{1}{2}\,\theta(m^B\,|\zeta|^2-3)\end{pmatrix}. \tag{39}$$

4. Any function $f \in D(\mathscr{L})$ can be written as $f = q_f + w_f$ with $q_f \in \operatorname{Ker} \mathscr{L}$ and $w_f \in (\operatorname{Ker} \mathscr{L})^{\perp}$ and one has

$$\langle \mathscr{L} f, f \rangle \geqslant \gamma_1 \, \|(1+|\zeta|)^{\frac{1}{2}} w_f\|^2_{\mathscr{H}}. \tag{40}$$

Proof. The proof is a straightforward extension of the classical Grad construction (cf. refs. 3 and 13) and the emphasis is only put on the new points. The α component of $\mathscr{L} f$ ($\alpha \in \{A, B\}$) is

$$\begin{aligned} &\left[\rho^\alpha \iint M^\alpha(\zeta_*)\, B^{\alpha\alpha}\, d\omega\, d\zeta_* + \rho^\beta \iint M^\beta(\zeta_*)\, B^{\beta\alpha}\, d\omega\, d\zeta_*\right] f^\alpha \\ &\quad - \rho^\alpha \iint M^\alpha(\zeta_*)\, f^\alpha(\zeta')\, B^{\alpha\alpha}\, d\omega\, d\zeta_* - \rho^\beta \iint M^\beta(\zeta_*)\, f^\alpha(\zeta')\, B^{\beta\alpha}\, d\omega\, d\zeta_* \\ &\quad - \rho^\alpha (M^\alpha)^{-1}\, J^{\alpha\alpha}(M^\alpha f^\alpha, M^\alpha) - \rho^\beta (M^\alpha)^{-1}\, J^{\beta\alpha}(M^\beta f^\beta, M^\alpha). \end{aligned} \tag{41}$$

The first line in (41) is $\nu_\alpha(\zeta)\, f^\alpha$ and the relation (32) follows from the estimate:

$$\gamma_0^{\alpha\beta}(1+|\zeta|) \leqslant \iint M^\beta(\zeta_*)\, B^{\beta\alpha}\, d\omega\, d\zeta_* \leqslant \gamma_1^{\alpha\beta}(1+|\zeta|), \qquad 0 < \gamma_0^{\alpha\beta} < \gamma_1^{\alpha\beta}, \tag{42}$$

which is valid for any $\alpha, \beta \in \{A, B\}$ (cf. ref. 13). The second and the third lines in (41) are $\mathscr{K}^\alpha(f^\alpha, f^\beta)$ and the operator $\mathscr{K}$

$$\begin{aligned} \mathscr{K}^\alpha \colon (f^\alpha, f^\beta) \mapsto & -\rho^\alpha \iint M^\alpha(\zeta_*)\, f^\alpha(\zeta')\, B^{\alpha\alpha}\, d\omega\, d\zeta_* \\ & -\rho^\beta \iint M^\beta(\zeta_*)\, f^\alpha(\zeta')\, B^{\beta\alpha}\, d\omega\, d\zeta_* \\ & -(M^\alpha)^{-1}\, [\rho^\alpha J^{\alpha\alpha}(M^\alpha f^\alpha, M^\alpha) + \rho^\beta J^{\beta\alpha}(M^\beta f^\beta, M^\alpha)] \end{aligned}$$

is compact.

Then the fact that $\mathscr{L}$ is self-adjoint in $\mathscr{H}$ follows by inspection. To obtain the formula (35)–(38) one plugs in the formula (11) $F^A = \rho^A M^A(1+\eta f^A)$, $F^B = \rho^B M^B(1+\eta f^B)$, $G^A = f^A$ and $G^B = f^B$, and retains the terms of order 1 with respect to η. A basic consequence of these formulas is the fact that the operator $\mathscr{L}$ is non-negative and that $\operatorname{Ker} \mathscr{L}$ coincides with the space of functions $f = (f^A, f^B)$ such that

$$f^A(\zeta'_*) + f^A(\zeta') - f^A(\zeta_*) - f^A(\zeta) \equiv 0, \tag{43}$$

$$f^B(\zeta'_*) + f^B(\zeta') - f^B(\zeta_*) - f^B(\zeta) \equiv 0, \tag{44}$$

and

$$f^A(\zeta'_*) + f^B(\zeta') - f^A(\zeta_*) - f^B(\zeta) \equiv 0. \tag{45}$$

From Eqs. (43) and (44) one deduces that for $f = (f^A, f^B) \in \text{Ker}\,\mathscr{L}$, f^A and f^B are contained in the five dimensional spaces defined by the formulas:

$$f^A(\zeta) = n^A + m^A u^A \cdot \zeta + \frac{\theta^A}{2}(m^A |\zeta|^2 - 3), \tag{46}$$

$$f^B(\zeta) = n^B + m^B u^B \cdot \zeta + \frac{\theta^B}{2}(m^B |\zeta|^2 - 3). \tag{47}$$

Then the relation (45) implies that $u^A = u^B$ and $\theta^A = \theta^B$. This completes the proof of point 3 of Proposition 2.2.

Since the operator $\mathscr{K}$ is compact, 0 is an isolated point in the spectra of the positive self-adjoint operator $\mathscr{L}$ and one has

$$\langle \mathscr{L} f, f \rangle \geqslant \delta_0 \, \|f\|^2_{\mathscr{H}} \qquad \text{for} \quad \forall f \in (\text{Ker}\,\mathscr{L})^{\perp}. \tag{48}$$

Furthermore with (32) one has for every $f \in D(\mathscr{L})$

$$\langle \mathscr{L} f, f \rangle \geqslant \delta \, \|(1+|\zeta|)^{\frac{1}{2}} f\|^2_{\mathscr{H}} - \|\mathscr{K}\| \, \|f\|^2_{\mathscr{H}}. \tag{49}$$

Writing $f = q_f + w_f$ one deduces (40) from (48) and (49). ∎

3. MILNE PROBLEM

The Milne problem is the analysis of the solution of the half-space problem in $\mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta$

$$\zeta_1 \, \partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta \tag{50}$$

with given incoming data at the point $x_1 = 0$. With the analysis of the previous section the results of ref. 1 are directly adapted leading to the following:

Theorem 3.1. Assume that the function $g(\zeta) = (g^A(\zeta), g^B(\zeta))$ given for $\zeta_1 > 0$ satisfies the estimate

$$\rho^A \int_{\zeta_1 > 0} (1+|\zeta|) \, |g^A(\zeta)|^2 \, M^A(\zeta) \, d\zeta$$
$$+ \rho^B \int_{\zeta_1 > 0} (1+|\zeta|) \, |g^B(\zeta)|^2 \, M^B(\zeta) \, d\zeta < \infty. \tag{51}$$

Then there exists a unique solution $f(x_1, \zeta)$ of the problem (50) with $|\zeta|^{\frac{1}{2}} f(x_1, \zeta) \in L^\infty(\mathbb{R}^+_{x_1}; \mathscr{H})$ and $\partial_{x_1} f(x_1, \zeta) \in L^2_{\text{loc}}(\mathbb{R}^+_{x_1}; \mathscr{H})$ which satisfies the relations:

$$f(0, \zeta) = g(\zeta) \qquad \text{for} \quad \zeta_1 > 0 \tag{52}$$

and

$$\langle \zeta_1 f \rangle = \begin{pmatrix} \rho^A \int \zeta_1 f^A(x_1, \zeta)\, M^A(\zeta)\, d\zeta \\ \rho^B \int \zeta_1 f^B(x_1, \zeta)\, M^B(\zeta)\, d\zeta \end{pmatrix} = 0. \tag{53}$$

Furthermore for x_1 going to infinity this solution converges exponentially to a "hydrodynamic" state q_f^∞:

$$\|(1+|\zeta|)^{\frac{1}{2}} (f - q_f^\infty)\|^2_{\mathscr{H}} \leqslant C e^{-\sigma x_1} \qquad \text{with} \quad \sigma < 2\gamma_1, \tag{54}$$

where

$$q_f^\infty = \begin{pmatrix} n_g^A + m^A(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\,\theta_g(m^A\,|\zeta|^2 - 3) \\ n_g^B + m^B(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\,\theta_g(m^B\,|\zeta|^2 - 3) \end{pmatrix}. \tag{55}$$

Proof. The proof follows the arguments of ref. 1 and just the main elements and the way to adapt them to the present situation are given below. We denote by D the set of functions f such that $|\zeta|^{\frac{1}{2}} f(x_1, \zeta) \in L^\infty(\mathbb{R}^+_{x_1}; \mathscr{H})$ and $\partial_{x_1} f(x_1, \zeta) \in L^2_{\text{loc}}(\mathbb{R}^+_{x_1}; \mathscr{H})$ which are solutions of the equation:

$$\zeta_1\, \partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta \tag{56}$$

with zero mean flux

$$\langle \zeta_1 f \rangle = \begin{pmatrix} \rho^A \int \zeta_1 f^A(x_1, \zeta)\, M^A(\zeta)\, d\zeta \\ \rho^B \int \zeta_1 f^B(x_1, \zeta)\, M^B(\zeta)\, d\zeta \end{pmatrix} = 0, \tag{57}$$

and decompose f into its hydrodynamic and kinetic parts:

$$f = q_f + w_f, \qquad q_f \in \operatorname{Ker} \mathscr{L}, \qquad w_f \in \operatorname{Ker} \mathscr{L}^\perp. \tag{58}$$

Multiplying (for the $\mathscr{H}$ scalar product) Eq. (56) by f, integrating first from 0 to X, and then letting $X \to \infty$ give, with (40), for any $f \in D$ the estimate:

$$\int_0^\infty \|(1+|\zeta|)^{\frac{1}{2}}\, w_f\|^2_{\mathscr{H}}\, dx_1 < \infty. \tag{59}$$

Next observe that for $\Xi \in \text{Ker}\,\mathscr{L}$ such that $\langle \zeta_1 \Xi \rangle = 0$ one has

$$0 = \partial_{x_1} \langle \zeta_1 \Xi(\zeta), f \rangle = \partial_{x_1} \langle \zeta_1 \Xi(\zeta), q_f + w_f \rangle. \tag{60}$$

Then for parity reason with (57) one has (no difference with the scalar case of ref. 1)

$$\langle \zeta_1 \Xi(\zeta), q_f \rangle \equiv 0, \tag{61}$$

and Eq. (60) is reduced to

$$\partial_{x_1} \langle \zeta_1 \Xi(\zeta), w_f \rangle \equiv 0. \tag{62}$$

With (59) this implies that

$$\langle \zeta_1 \Xi(\zeta), w_f \rangle \equiv 0. \tag{63}$$

Therefore one comes with the relation

$$\langle \zeta_1 f, f \rangle = \langle \zeta_1 w_f, w_f \rangle \tag{64}$$

and finally obtains:

$$\tfrac{1}{2}\, \partial_{x_1} \langle \zeta_1 w_f, w_f \rangle + \langle \mathscr{L} w_f, w_f \rangle = 0. \tag{65}$$

This implies that the quantity $\langle \zeta_1 w_f, w_f \rangle$ is non-negative and decays monotonically to zero because $\mathscr{L}$ is non-negative.

For the uniqueness of the solution one uses the linearity and considers a solution that vanishes for $x_1 = 0$ and $\zeta_1 > 0$:

$$f(0, \zeta) = \begin{pmatrix} f^A(0, \zeta) \\ f^B(0, \zeta) \end{pmatrix} = 0 \qquad \text{for} \quad \zeta_1 > 0. \tag{66}$$

From (65) one deduces for such solution the relation:

$$\int_0^\infty \|(1 + |\zeta|)^{\frac{1}{2}}\, w_f(x_1, \zeta)\|_{\mathscr{H}}^2 \, dx_1 = 0. \tag{67}$$

Therefore for such solution Eq. (50) becomes

$$\zeta_1 \, \partial_{x_1} q_f = 0, \tag{68}$$

where

$$q_f = \begin{pmatrix} n_0^A(x_1) + m^A[u_{2,0}(x_1)\,\zeta_2 + u_{3,0}(x_1)\,\zeta_3] + \frac{\theta_0(x_1)}{2}\,(m^A\,|\zeta|^2 - 3) \\ n_0^B(x_1) + m^B[u_{2,0}(x_1)\,\zeta_2 + u_{3,0}(x_1)\,\zeta_3] + \frac{\theta_0(x_1)}{2}\,(m^B\,|\zeta|^2 - 3) \end{pmatrix}. \tag{69}$$

As in ref. 1, p. 335 one multiplies α component of (68) by the vector

$$\zeta_1 \begin{pmatrix} 1 \\ \zeta_2 \\ \zeta_3 \\ |\zeta|^2 \end{pmatrix} M^\alpha(\zeta), \tag{70}$$

integrates over $\zeta_1 > 0$, and concludes that q_f is identically zero. This concludes the proof of uniqueness.

For the exponential integrated decay of w_f:

$$\int_0^\infty e^{\sigma x_1}\,\|(1+|\zeta|)^{\frac{1}{2}}\,w_f(x_1,\zeta)\|_{\mathscr{H}}^2\,dx_1 < \infty \qquad \text{with} \quad \sigma < 2\gamma_1, \tag{71}$$

one multiplies (for the $\mathscr{H}$ scalar product) Eq. (50) by $e^{\sigma x_1} f$, uses again the relation

$$\langle \zeta_1 f, f\rangle = \langle \zeta_1 w_f, w_f\rangle \tag{72}$$

and the estimate (40), and proceeds as in ref. 1. The pointwise decay of w_f is obtained along the same line and the exponential convergence of f to a hydrodynamic state:

$$q_f^\infty = \begin{pmatrix} n_g^A + m^A(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\,\theta_g(m^A\,|\zeta|^2 - 3) \\ n_g^B + m^B(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\,\theta_g(m^B\,|\zeta|^2 - 3) \end{pmatrix} \tag{73}$$

follows.

For the existence (and the numerical computation) of the solution, one starts with a problem in a finite slab $0 < x_1 < L$ and solves the problem (50) with the same incoming boundary condition at $x_1 = 0$ and a specular boundary condition at $x_1 = L$:

$$f^\alpha(L, \zeta_1, \zeta_2, \zeta_3) = f^\alpha(L, -\zeta_1, \zeta_2, \zeta_3) \qquad \text{for} \quad \alpha = (A, B). \tag{74}$$

To prove the existence of such solution a Fredholm alternative is used to reduce the problem to the fact that any solution of

$$\zeta_1\,\partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \{0 < x_1 < L\} \times \mathbb{R}^3_\zeta \tag{75}$$

with the boundary condition (74) and zero incoming boundary condition

$$f(0,\zeta)=0 \qquad \text{for} \quad \zeta_1>0 \tag{76}$$

is identically zero. Multiplying (for the $\mathscr{H}$ scalar product) Eq. (75) by f and integrating over $0<x_1<L$ give, with the specular boundary condition (74), the relation

$$\begin{aligned}\gamma_1 \int_0^L &\|(1+|\zeta|)^{\frac{1}{2}}\, w_f(x_1,\zeta)\|_{\mathscr{H}}^2\, dx_1\\ &\leqslant \int_0^L \langle \mathscr{L} f, f\rangle\, dx_1 = \tfrac{1}{2}\langle \zeta_1 f(0,\zeta), f(0,\zeta)\rangle\\ &\leqslant \tfrac{1}{2}\int_{\zeta_1>0} \rho^A M^A \zeta_1\, |f^A(0,\zeta)|^2\, d\zeta + \tfrac{1}{2}\int_{\zeta_1>0} \rho^B M^B \zeta_1\, |f^B(0,\zeta)|^2\, d\zeta\\ &=0.\end{aligned} \tag{77}$$

This implies that w_f is identically zero and the rest of the proof follows as in ref. 1.

Now denote by f_L the solution of the slab problem. From Eq. (75) multiplied in $\mathscr{H}$ by Ξ :

$$\partial_{x_1}\langle \zeta_1 \Xi, f_L\rangle = 0, \tag{78}$$

one deduces that any solution of (75) with boundary condition (74) satisfies

$$\langle \zeta_1 \Xi, f_L\rangle = 0. \tag{79}$$

This allows to adapt to the domain $]0,L[\times\mathbb{R}^3_\zeta$ (uniformly with respect to L) all the estimates that were derived above for the half space. Finally take the limit $L\to\infty$. ▮

Equation (73) defines a "continuous mapping" from the space of functions $g=(g^A,g^B)$ such that

$$\rho^A \int_{\zeta_1>0} (1+|\zeta|)\,|g^A(\zeta)|^2\, M^A\, d\zeta + \rho^B \int_{\zeta_1>0} (1+|\zeta|)\,|g^B(\zeta)|^2\, M^B\, d\zeta < \infty \tag{80}$$

into $\mathbb{R}^5$:

$$g \mapsto \{n_g^A, n_g^B, u_{2,g}, u_{3,g}, \theta_g\}. \tag{81}$$

From the uniqueness of the solution one deduces the following symmetry property:

Corollary 3.1. Assume that the incoming data $g(\zeta)=(g^A(\zeta), g^B(\zeta))$ are invariant under rotation with respect to the ζ_1 axis, i.e., they depend only on ζ_1 and $\sqrt{\zeta_2^2+\zeta_3^2}$. Then the same property holds for the solution of Milne problem:

$$|\zeta|^{\frac{1}{2}} f(x_1,\zeta) \in L^\infty(\mathbb{R}^+_{x_1}; \mathscr{H}), \qquad \partial_{x_1} f(x_1,\zeta) \in L^2_{\mathrm{loc}}(\mathbb{R}^+_{x_1}; \mathscr{H}), \tag{82}$$

$$\zeta_1\, \partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta, \tag{83}$$

$$f(0,\zeta) = g(\zeta), \tag{84}$$

$$\langle \zeta_1 f \rangle = \begin{pmatrix} \rho^A \int \zeta_1 f^A(x_1,\zeta)\, M^A(\zeta)\, d\zeta \\ \rho^B \int \zeta_1 f^B(x_1,\zeta)\, M^B(\zeta)\, d\zeta \end{pmatrix} = 0. \tag{85}$$

This property is often used in numerical computations of the Milne problem for the reduction of the number of independent variables. As is shown in the next section, the Kramers problem can be reduced to the Milne problem, so that the same symmetry property holds for the Kramers problem.

Example 3.1. As an example of application of Theorem 3.1 and Corollary 3.1, we take weak evaporation and condensation problem. Consider a binary mixture of vapors (species A and B) in contact with their plane condensed phase. Far from the interface the mixture is supposed to be in a uniform equilibrium state. If the temperature T_w of the condensed phase is different from that of the mixture, evaporation or condensation takes place at the interface. When the temperature difference is small, the problem can be formulated as follows:

$$\zeta_1 \partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta,$$

$$f(0,\zeta) = 0 \qquad \text{for} \quad \zeta_1 > 0,$$

$$f \to \begin{pmatrix} n^A + m^A(u_1\zeta_1 + u_2\zeta_2 + u_3\zeta_3) + \frac{1}{2}\,\theta(m^A\,|\zeta|^2 - 3) \\ n^B + m^B(u_1\zeta_1 + u_2\zeta_2 + u_3\zeta_3) + \frac{1}{2}\,\theta(m^B\,|\zeta|^2 - 3) \end{pmatrix} \qquad \text{as} \quad x_1 \to \infty.$$

Here n^A (or n^B) corresponds to the difference of the number density of species A (or B) between the mixture at infinity and the mixture in (phase) equilibrium with the condensed phase at temperature T_w, θ to the difference of the temperature of the mixture at infinity from T_w, and u_i to the flow velocity of the mixture at infinity. Note that n^A, n^B, θ, and u_i are constants. Now consider the function $f - (m^A\zeta_1, m^B\zeta_1)\, u_1$. The function satisfies the estimate (52) and the relation (53) in Theorem 3.1, and the theorem can be applied. Then one finds that the solution of the present problem f exists and the constants n^A, n^B, θ, u_2, and u_3 are unique for u_1. From Corollary 3.1 and the linearity of the problem one further finds that the following relation holds when steady evaporation or condensation takes place:

$$u_2 = u_3 = 0, \qquad \begin{pmatrix} n^A \\ n^B \\ \theta \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} u_1, \qquad (c_i\text{: constant}).$$

With the linearity of the problem and the fact that the x_1 independent hydrodynamic states are solutions of the Milne problem, the Theorem 3.1 can be rephrased as follows:

Theorem 3.2. For any function $g(\zeta) = (g^A(\zeta), g^B(\zeta))$ given for $\zeta_1 > 0$ which satisfies the estimate

$$\rho^A \int_{\zeta_1 > 0} (1 + |\zeta|)\, |g^A(\zeta)|^2\, M^A(\zeta)\, d\zeta + \rho^B \int_{\zeta_1 > 0} (1 + |\zeta|)\, |g^B(\zeta)|^2\, M^B(\zeta)\, d\zeta < \infty, \tag{86}$$

there exists a unique set of values $\{n_g^A, n_g^B, u_{2,g}, u_{3,g}, \theta_g\}$ such that the solution of the Milne problem with incoming data:

$$\begin{pmatrix} g^A \\ g^B \end{pmatrix} - \begin{pmatrix} n_g^A + m^A(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\theta_g(m^A|\zeta|^2 - 3) \\ n_g^B + m^B(u_{2,g}\zeta_2 + u_{3,g}\zeta_3) + \frac{1}{2}\theta_g(m^B|\zeta|^2 - 3) \end{pmatrix} \tag{87}$$

and zero mean flux converges exponentially to zero.

The solution of the Milne problem with incoming data given by (87) is called the Knudsen-layer solution of the half-space problem with incoming data g.

Remark 3.1. Theorem 3.2 is the theorem which was conjectured and used in ref. 17. The correspondence between the notation in Appendix B of ref. 17 and the present one is as follows:

$$(\phi^A, \phi^B) \leftrightarrow (f^A, f^B),$$

$$\sum_{\beta = A, B} K^{\beta\alpha} C^{\beta} \tilde{L}^{\beta\alpha}(\phi^\beta, \phi^\alpha) \leftrightarrow -\mathscr{L}(f),$$

$$(\hat{m}^A, \hat{m}^B) \leftrightarrow (m^A, m^B),$$

$$(c^A + \tfrac{3}{2}c_4, c^B + \tfrac{3}{2}c_4, c_2, c_3, c_4) \leftrightarrow -(n_g^A, n_g^B, u_{2,g}, u_{3,g}, \theta_g).$$

4. KRAMERS PROBLEM

The Kramers problem is to solve the Boltzmann equation (50) in the half space $\mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta$, with given incoming data at $x_1 = 0$ and a distribution growing linearly at $x_1 = \infty$ in the following way:

$$\lim_{x_1\to\infty} \frac{d}{dx_1}\frac{1}{3}\int \rho^A m^A \zeta^2 f^A M^A\, d\zeta = -\lim_{x_1\to\infty} \frac{d}{dx_1}\frac{1}{3}\int \rho^B m^B \zeta^2 f^B M^B\, d\zeta = p^A, \quad (88)$$

$$\lim_{x_1\to\infty} \frac{d}{dx_1}\left(\int \rho^A m^A \zeta_2 f^A M^A\, d\zeta + \int \rho^B m^B \zeta_2 f^B M^B\, d\zeta\right) = u_2, \quad (89)$$

$$\lim_{x_1\to\infty} \frac{d}{dx_1}\left(\int \rho^A m^A \zeta_3 f^A M^A\, d\zeta + \int \rho^B m^B \zeta_3 f^B M^B\, d\zeta\right) = u_3, \quad (90)$$

$$\lim_{x_1\to\infty} \frac{d}{dx_1}\left(\frac{1}{3}\int \rho^A (m^A\zeta^2 - 3) f^A M^A\, d\zeta + \frac{1}{3}\int \rho^B (m^B\zeta^2 - 3) f^B M^B\, d\zeta\right) = \theta. \quad (91)$$

The problem can be reduced to the Milne problem as follows. Consider the function of the form $f_0 + x_1 f_1$ with f_0 and f_1 independent of x_1 which satisfies the above condition at infinity. Then f_0 and f_1 satisfy

$$\mathscr{L} f_1 = 0, \quad (92)$$

$$\mathscr{L} f_0 = -\zeta_1 f_1. \quad (93)$$

These equations imply that $f_1 \in \operatorname{Ker} \mathscr{L}$ and $\zeta_1 f_1 \in (\operatorname{Ker} \mathscr{L})^\perp$. From these and the conditions at infinity one finds that

$$f_1^A = p^A + m^A u_2^A \zeta_2 + m^A u_3^A \zeta_3 + \tfrac{1}{2}\theta(m^A\zeta^2 - 5), \quad (94)$$

$$f_1^B = -p^A + m^B u_2^B \zeta_2 + m^B u_3^B \zeta_3 + \tfrac{1}{2}\theta(m^B\zeta^2 - 5). \quad (95)$$

Then there is a unique solution $f_0 \in (\operatorname{Ker} \mathscr{L})^\perp$ with $\mathscr{L} f_0 = -\zeta_1 f_1$. The Kramers problem for f can be reduced to the Milne problem for $f - (f_0 + x_1 f_1)$.

Remark 4.1. The coefficients p^A, u_2, u_3, and θ are, respectively, the gradients of the partial pressure of component gas A, flow velocities of the mixture in the x_2- and x_3-directions, and temperature of the mixture. The first equality in (88) implies that the pressure of the total mixture is uniform far from the boundary. Note that, for a single-component gas, the Kramers problem is well-posed for linearly growing flow velocities in x_2 and x_3-directions and temperature, but not for linearly growing pressure.

5. MILNE PROBLEM WITH SOURCE TERM

This section is devoted to the analysis of the half-space problem with an internal source term $h(x_1, \zeta) = (h^A(x_1, \zeta), h^B(x_1, \zeta))$

$$\zeta_1\, \partial_{x_1} f + \mathscr{L} f = h \qquad \text{in} \quad \mathbb{R}^+_{x_1} \times \mathbb{R}^3_\zeta \tag{96}$$

with given incoming data at $x_1 = 0$.

Since the problem is linear the issue of uniqueness concerns solutions with $h \equiv 0$ and therefore this is completely covered by the results of previous sections. Only existence and asymptotic behavior have to be considered. The proof follows the line of the Section 8 in ref. 1. Using once more the linearity one observes that it is enough to consider the homogeneous boundary-value problem (i.e., $g(\zeta) \sim 0$) and the following two cases:

$$\forall x_1 > 0 \quad h_1(x_1, .) \in (\text{Ker } \mathscr{L})^{\perp} \tag{97}$$

and

$$\forall x_1 > 0 \quad h_2(x_1, .) \in \text{Ker } \mathscr{L}. \tag{98}$$

In the first case one starts as in Section 3 from the solution in a finite slab $0 < x_1 < L$ of

$$\zeta_1\, \partial_{x_1} f + \mathscr{L} f = h_1 \qquad \text{in} \quad \{0 < x_1 < L\} \times \mathbb{R}^3_\zeta \tag{99}$$

with zero incoming boundary condition at $x_1 = 0$ and a specular boundary condition at $x_1 = L$. As above one writes $f = q_f + w_f$ and from the specular boundary condition one deduces, for any $\Xi(\zeta) \in \text{Ker } \mathscr{L}$ such that $\langle \zeta_1 \Xi \rangle = 0$, the relations:

$$\langle \zeta_1 \Xi(\zeta), q_f \rangle \equiv 0, \tag{100}$$

$$\partial_{x_1} \langle \zeta_1 \Xi(\zeta), w_f \rangle \equiv 0, \tag{101}$$

$$\langle \zeta_1 \Xi(\zeta), w_f \rangle \equiv 0, \tag{102}$$

and finally

$$\langle \zeta_1 f, f \rangle = \langle \zeta_1 w_f, w_f \rangle. \tag{103}$$

In particular the solution satisfies in the slab the relation:

$$\langle \zeta_1 f \rangle \equiv 0. \tag{104}$$

Multiplying Eq. (99) by $e^{\gamma x_1} f$ gives

$$\tfrac{1}{2}\,\partial_{x_1} e^{\gamma x_1}\langle \zeta_1 w_f, w_f\rangle - \frac{\gamma}{2}\, e^{\gamma x_1}\langle \zeta_1 w_f, w_f\rangle + e^{\gamma x_1}\langle \mathscr{L} w_f, w_f\rangle = \langle h_1, e^{\gamma x_1} f\rangle. \tag{105}$$

Since h_1 belongs to $(\mathrm{Ker}\,\mathscr{L})^{\perp}$ one has:

$$\langle h_1, e^{\gamma x_1} f\rangle = \langle h_1, e^{\gamma x_1} w_f\rangle. \tag{106}$$

Then with the Cauchy–Schwarz estimate and for $\gamma_2 < 2\gamma_1$ and small enough one obtains, with an L independent constant C, the relation:

$$\int_0^L e^{\gamma_2 x_1}\, \|(1+|\zeta|)^{\frac{1}{2}}\, w_f(x_1,\zeta)\|^2_{\mathscr{H}}\, dx_1 \leqslant C \int_0^L e^{\gamma_2 x_1}\, \|(1+|\zeta|)^{\frac{1}{2}}\, h_1(x_1,\zeta)\|^2_{\mathscr{H}}\, dx_1 \tag{107}$$

and continuing as in the Section 3 one proves the following proposition:

Proposition 5.1. Assume that

$$h_1(x_1,\zeta) \in L^{\infty}(\mathbb{R}^+_{x_1}, (\mathrm{Ker}\,\mathscr{L})^{\perp}) \tag{108}$$

and for some $\gamma > 0$

$$\int_0^{\infty} e^{\gamma x_1} \|(1+|\zeta|)^{\frac{1}{2}}\, h_1(x_1,\zeta)\|^2_{\mathscr{H}}\, dx_1 < \infty. \tag{109}$$

Then there exists a unique "bounded" solution f of the Milne problem with source h_1:

$$\zeta_1\, \partial_{x_1} f + \mathscr{L} f = h_1, \quad f(0,\zeta) = 0 \qquad \text{for} \quad \zeta_1 > 0. \tag{110}$$

This solution satisfies the relation

$$\langle \zeta_1 f\rangle = 0 \tag{111}$$

and for $0 < \gamma_2$ small enough the estimate

$$\int_0^{\infty} e^{\gamma_2 x_1}\, \|(1+|\zeta|)^{\frac{1}{2}}\, w_f(x_1,\zeta)\|^2_{\mathscr{H}}\, dx_1 < \infty. \tag{112}$$

And finally it converges exponentially to a hydrodynamic state:

$$q_f^{\infty} = \begin{pmatrix} n_0^A + m^A(u_{2,0}\zeta_2 + u_{3,0}\zeta_3) + \frac{1}{2}\,\theta_0(m^A\,|\zeta|^2 - 3) \\ n_0^B + m^B(u_{2,0}\zeta_2 + u_{3,0}\zeta_3) + \frac{1}{2}\,\theta_0(m^B\,|\zeta|^2 - 3) \end{pmatrix}. \tag{113}$$

Now assume that $h = h_2$ is an x_1 dependent hydrodynamic state:

$$h_2(x_1, \zeta) = \begin{pmatrix} n^A(x_1) \\ n^B(x_1) \end{pmatrix} + u_1(x_1)\,\zeta_1 \begin{pmatrix} m^A \\ m^B \end{pmatrix} + u_2(x_1)\,\zeta_2 \begin{pmatrix} m^A \\ m^B \end{pmatrix} + u_3(x_1)\,\zeta_3 \begin{pmatrix} m^A \\ m^B \end{pmatrix} + \frac{1}{2}\,\theta(x_1) \begin{pmatrix} m^A\,|\zeta|^2 - 3 \\ m^B\,|\zeta|^2 - 3 \end{pmatrix} \tag{114}$$

which also satisfies for some $\gamma > 0$ the hypothesis

$$\int_0^\infty e^{\gamma x_1}\, \|(1+|\zeta|)^{\frac{1}{2}}\, h_2\,(x_1, \zeta)\|_{\mathscr{H}}^2\, dx_1 < \infty. \tag{115}$$

In order to reduce the problem to a problem with a source in $(\mathrm{Ker}\,\mathscr{L})^\perp$ one introduces a function f_0 such that

$$-\zeta_1\,\partial_{x_1} f_0 + h_2 \in (\mathrm{Ker}\,\mathscr{L})^\perp \tag{116}$$

and

$$\int_0^\infty e^{\gamma x_1}\, \|(1+|\zeta|)^{\frac{1}{2}}\,(-\zeta_1\,\partial_{x_1} f_0 + \mathscr{L}(f_0) + h_2)\|_{\mathscr{H}}^2\, dx_1 < \infty. \tag{117}$$

When this is realized one observes that

$$-(\zeta_1\,\partial_{x_1} f_0 + \mathscr{L} f_0) + h_2 \in (\mathrm{Ker}\,\mathscr{L})^\perp. \tag{118}$$

Therefore according to Proposition 5.1 and Theorem 3.1 there exists a unique solution f_1 of the problem

$$\zeta_1\,\partial_{x_1} f_1 + \mathscr{L} f_1 = -(\zeta_1\,\partial_{x_1} f_0 + \mathscr{L} f_0) + h_2 \tag{119}$$

with

$$f_1(0, \zeta) = -f_0(0, \zeta) \qquad \text{for} \quad \zeta_1 > 0, \tag{120}$$

$$\langle \zeta_1 f_1 \rangle = 0, \tag{121}$$

and one obtains:

$$\zeta_1 \partial_{x_1}(f_1 + f_0) + \mathscr{L}(f_1 + f_0) = h_2 \qquad \text{for} \quad x_1 > 0. \tag{122}$$

To realize (116) and (117) one looks for solutions of the form:

$$f_0(x_1,\zeta)=a(x_1)\begin{pmatrix} m^A \\ m^B \end{pmatrix}+\zeta_1\begin{pmatrix} m^A U^A(x_1) \\ m^B U^B(x_1) \end{pmatrix}+U_2(x_1)\,\zeta_1\zeta_2\begin{pmatrix} (m^A)^2 \\ (m^B)^2 \end{pmatrix}$$
$$+U_3(x_1)\,\zeta_1\zeta_3\begin{pmatrix} (m^A)^2 \\ (m^B)^2 \end{pmatrix}+D(x_1)\,\zeta_1\,|\zeta|^2\begin{pmatrix} (m^A)^2 \\ (m^B)^2 \end{pmatrix}. \tag{123}$$

The relation (116) is equivalent to:

$$\langle \zeta_1\,\partial_{x_1} f_0, \Xi\rangle = \langle h_2, \Xi\rangle, \qquad \forall \Xi \in \operatorname{Ker}\mathscr{L}, \tag{124}$$

which is equivalent to the following system of 6 linear equations:

$$\begin{aligned} a'(x_1) &= u_1(x_1),\\ U_2'(x_1) &= u_2(x_1),\\ U_3'(x_1) &= u_3(x_1), \end{aligned} \tag{125}$$

and

$$(U^A)'\,(x_1)+5D'(x_1)=n^A(x_1),$$
$$(U^B)'\,(x_1)+5D'(x_1)=n^B(x_1),$$
$$\frac{5\rho^A}{(m^A)^{\frac{3}{2}}}(U^A)'\,(x_1)+\frac{5\rho^B}{(m^B)^{\frac{3}{2}}}(U^B)'\,(x_1)+35\left(\frac{\rho^A}{(m^A)^{\frac{3}{2}}}+\frac{\rho^B}{(m^B)^{\frac{3}{2}}}\right)D'(x_1)$$
$$=3\left\{\left(\frac{\rho^A}{(m^A)^{\frac{3}{2}}}+\frac{\rho^B}{(m^B)^{\frac{3}{2}}}\right)\theta(x_1)+\frac{\rho^A}{(m^A)^{\frac{3}{2}}}n^A(x_1)+\frac{\rho^B}{(m^B)^{\frac{3}{2}}}n^B(x_1)\right\}. \tag{126}$$

Because of the hypothesis (115), the integration of (125) and (126) from x_1 to ∞ leads to the convergence of the functions $a(x_1)$, $U^A(x_1)$, $U^B(x_1)$, $U_2(x_1)$, $U_3(x_1)$, and $D(x_1)$ to some constants for $x_1\to\infty$. One can choose these constants as zero because their values do not influence the relation (124). Then the solution of (125) is explicitly given by

$$\begin{aligned} a(x_1) &= -\int_{x_1}^{\infty} u_1(s)\,ds,\\ U_2(x_1) &= -\int_{x_1}^{\infty} u_2(s)\,ds,\\ U_3(x_1) &= -\int_{x_1}^{\infty} u_3(s)\,ds, \end{aligned} \tag{127}$$

and, observing that the determinant of (126) is

$$10\left(\frac{\rho^A}{(m^A)^{\frac{3}{2}}}+\frac{\rho^B}{(m^B)^{\frac{3}{2}}}\right),$$

the functions $U^A(x_1)$, $U^B(x_1)$, and $D(x_1)$ are by linear sums of the integrals:

$$-\int_{x_1}^{\infty} n^A(s)\,ds, \qquad -\int_{x_1}^{\infty} n^B(s)\,ds, \qquad \text{and} \qquad -\int_{x_1}^{\infty} \theta(s)\,ds.$$

Thus the hypothesis (115) implies that $f_0(x_1,\zeta)$ converges exponentially fast to zero for $x_1 \to \infty$. Finally, using the fact that $(m^A\zeta_1, m^B\zeta_1)$ is a solution of the linearized Boltzmann equation (without source term), with the above construction we have obtained the following:

Theorem 5.1. Assume that the data g and h satisfy the hypothesis

$$\rho^A\int_{\zeta_1>0}(1+|\zeta|)\,|g^A(\zeta)|^2\,M^A\,d\zeta+\rho^B\int_{\zeta_1>0}(1+|\zeta|)\,|g^B(\zeta)|^2\,M^B\,d\zeta<\infty \tag{128}$$

and for some $\gamma>0$

$$\int_0^{\infty} e^{\gamma x_1}\,\|(1+|\zeta|)^{\frac{1}{2}}\,h(x_1,\zeta)\|_{\mathscr{H}}^2\,dx_1<\infty. \tag{129}$$

Then there exists a unique function f with $|\zeta|^{\frac{1}{2}}\,f(x_1,\zeta)\in L^\infty(\mathbb{R}_{x_1}^+;\mathscr{H})$ which is a solution of the problem:

$$\zeta_1\,\partial_{x_1} f+\mathscr{L}f=h \qquad \text{in} \quad \mathbb{R}_{x_1}^+\times\mathbb{R}_\zeta^3, \tag{130}$$

$$f(0,\zeta)=g(0,\zeta) \qquad \text{for} \quad \zeta_1>0, \tag{131}$$

$$\left\langle\begin{pmatrix} m^A \\ m^B\end{pmatrix}\zeta_1, f\right\rangle=0 \qquad \text{at} \quad x_1=0. \tag{132}$$

This solution converges exponentially fast to a hydrodynamic set of the following form:

$$q_f^\infty=\begin{pmatrix} n^A+m^A(u_1\zeta_1+u_2\zeta_2+u_3\zeta_3)+\frac{1}{2}\,\theta(m^A\,|\zeta|^2-3) \\ n^B+m^B(u_1\zeta_1+u_2\zeta_2+u_3\zeta_3)+\frac{1}{2}\,\theta(m^B\,|\zeta|^2-3)\end{pmatrix} \tag{133}$$

with

$$u_1 = \left(\frac{\rho^A}{(m^A)^{\frac{1}{2}}} + \frac{\rho^B}{(m^B)^{\frac{1}{2}}}\right)^{-1} \int_0^\infty \left\langle \begin{pmatrix} (m^A)^2 \\ (m^B)^2 \end{pmatrix}, h \right\rangle dx_1. \tag{134}$$

In particular when h is a pure non-hydrodynamic set

$$\forall x_1 > 0 \qquad h(x_1, \zeta) \in (\mathrm{Ker}\, \mathscr{L})^{\perp}, \tag{135}$$

the u_1 vanishes in (133) and the flux of each component gas is equivalent to zero:

$$\langle \zeta_1 f \rangle = 0. \tag{136}$$

Remark 5.1. In contrast to Theorem 3.1, specified is not the flux $\langle \zeta_1 f \rangle$ of each component gas but the flux $\langle \binom{m^A}{m^B} \zeta_1, f \rangle$ of total mixture at $x_1 = 0$. Note that the latter as well as the former is in general not conservative because of the source term h. Specifying the flux of total mixture, not the flux of each component gas, is due to the difference between $U^A(0)$ and $U^B(0)$ in f_0.

Remark 5.2. The Milne problem with a source term occurs when one considers the higher orders of the Knudsen number in the analysis of the Knudsen layer.[13] For instance, when the Reynolds number is small (i.e., of the order of the Knudsen number), the source term comes from the curvature of boundary.[10] When the Reynolds number is $O(1)$, there are two types of origins of the source term: one is the curvature of boundary and the other is the interaction between the velocity distributions in the lower orders of the Knudsen number through the Boltzmann collision integrals.[11]

6. SOME ESTIMATES RELATED TO SLAB PROBLEM

The solution f_L in the bounded domain $]0, L[\times \mathbb{R}^3_\zeta$ with the reflection condition which was introduced in the proof of Theorem 3.1 is also used for numerical computation. Therefore below are given some refined estimates concerning its convergence to the solution of the Milne problem. Such estimate in the case of the one component linearized Boltzmann equation has already been obtained by Peralta;[9] in fact the present section is an extension of his approach to the multicomponent case.

First we prove

Lemma 6.1. The solution f_L is bounded (uniformly with respect to L) in the space $L^\infty([\delta, L]; \mathscr{H})$ $(\delta > 0)$.

Proof. The proof follows from a uniform estimate on $(1+|\zeta|)^{\frac{1}{2}} \partial_{x_1} w_{f_L}$ in $L^2(]\delta, L[; \mathscr{H})$. First introduce a smooth scalar function $\phi(x_1)$ equal to 1 for $x > \delta$ and zero near $x_1 = 0$ and consider the equation

$$\zeta_1 \, \partial_{x_1}^2 (\phi f_L) + \mathscr{L} \, \partial_{x_1}(\phi f_L) = \zeta_1 \, \partial_{x_1}(\phi'(x_1) \, f_L). \tag{137}$$

Multiply in $\mathscr{H}$ (137) by $\partial_{x_1}(\phi f_L)$ and integrate over $[0, L]$ to obtain

$$\gamma_1 \int_0^L \|(1+|\zeta|)^{\frac{1}{2}} w_{\partial_{x_1}(\phi f_L)}\|_{\mathscr{H}}^2 \, dx_1$$

$$\leqslant \int_0^L \langle \zeta_1 w_{\partial_{x_1}(\phi' f_L)}, w_{\partial_{x_1}(\phi f_L)} \rangle \, dx_1 - \tfrac{1}{2} \langle \zeta_1 \partial_{x_1} f_L, \partial_{x_1} f_L \rangle (L). \tag{138}$$

With the specular reflection at $x_1 = L$ and the Galilean invariance of the collision operator, one has

$$(\mathscr{L} f_L)(L, \zeta_1, \zeta_2, \zeta_3) = (\mathscr{L} f_L)(L, -\zeta_1, \zeta_2, \zeta_3) \tag{139}$$

and from the equation one deduces the relation:

$$(\partial_{x_1} f_L)(L, \zeta_1, \zeta_2, \zeta_3) = -(\partial_{x_1} f_L)(L, -\zeta_1, \zeta_2, \zeta_3). \tag{140}$$

Therefore in (138) one has

$$\langle \zeta_1 \partial_{x_1} f_L, \partial_{x_1} f_L \rangle (L) = 0 \tag{141}$$

and the proof of the lemma can be completed by standard Cauchy–Schwarz estimates. ▮

With the above lemma one has

Proposition 6.1. Let $g = (g^A, g^B)$ the incoming data and denote by $f \in D$ the corresponding solution of the half-space problem (with zero mean flux) and by f_L the solution of the equation

$$\zeta_1 \, \partial_{x_1} f + \mathscr{L} f = 0 \qquad \text{in} \quad \{0 < x_1 < L\} \times \mathbb{R}_\zeta^3 \tag{142}$$

with the same incoming data and specular reflection condition at $x_1 = L$. Then f_L converges exponentially to f. More precisely, for two positive constants $a < \gamma_1$ and $d < \gamma_1 - a$,

$$\frac{1}{2}\, \| |\zeta_1|^{\frac{1}{2}} [f(0,\zeta) - f_L(0,\zeta)] \|_{\mathscr{H}}^2 + \left(\gamma_1 - \frac{a}{2}\right) \int_0^L e^{a x_1}\, \| (1+|\zeta|)^{\frac{1}{2}}\, w_{(f-f_L)} \|_{\mathscr{H}}^2\, dx_1 \leqslant C e^{-dL}. \tag{143}$$

Proof. Let us denote by r the function $f - f_L$. Introduce the equation for $r e^{a x_1/2}$ and multiply in $\mathscr{H}$ this equation by $r e^{a x_1/2}$ to obtain

$$-\frac{1}{2} \langle \zeta_1 r(0,\zeta), r(0,\zeta) \rangle + \left(\gamma_1 - \frac{a}{2}\right) \int_0^L e^{a x_1}\, \| (1+|\zeta|)^{\frac{1}{2}}\, w_r \|_{\mathscr{H}}^2\, dx_1 \leqslant \frac{1}{2} |\langle \zeta_1 r(L,\zeta), r(L,\zeta) \rangle|\, e^{a x_1}. \tag{144}$$

Observe that on one hand $r(0,\zeta)$ is zero for $\zeta_1 > 0$, which gives

$$-\tfrac{1}{2} \langle \zeta_1 r(0,\zeta), r(0,\zeta) \rangle = \tfrac{1}{2} \langle |\zeta_1|\, r(0,\zeta), r(0,\zeta) \rangle. \tag{145}$$

Then on the other hand notice that the function r satisfies the same orthogonality properties as the functions f and f_L. This implies the relation:

$$\begin{aligned} |\langle \zeta_1 r(L,\zeta), r(L,\zeta) \rangle| &= |\langle \zeta_1 w_r(L,\zeta), w_r(L,\zeta) \rangle| \\ &\leqslant |\langle \zeta_1 w_f(L,\zeta), w_f(L,\zeta) \rangle| + 2\, |\langle \zeta_1 w_{f_L}(L,\zeta), w_f(L,\zeta) \rangle| \\ &\quad + |\langle \zeta_1 w_{f_L}(L,\zeta), w_{f_L}(L,\zeta) \rangle|. \end{aligned} \tag{146}$$

With the specular reflection one has

$$\langle \zeta_1 w_{f_L}(L,\zeta), w_{f_L}(L,\zeta) \rangle = 0 \tag{147}$$

and therefore (143) follows from the uniform estimate on $w_{f_L}(L,\zeta)$ (Lemma 6.1) and from the exponential decay of $w_f(L,\zeta)$. ∎

Corollary 6.1. The solution with specular reflection at $x_1 = L$ gives an exponential estimate of the asymptotic for the hydrodynamic limit q_f^∞ [cf. (73)] of the solution of the Milne problem according to the formula:

$$\| q_f^\infty - q_{f_L}(L,\zeta) \|_{\mathscr{H}} \leqslant C' e^{-dL}. \tag{148}$$

Proof. Integrate over $[0, L]$ the equation for r multiplied in $\mathscr{H}$ by $\Xi\chi_{\{\zeta_1>0\}}$. Here $\chi_{\{\zeta_1>0\}}$ is the characteristic function of $\{\zeta_1>0\}$. Then use the estimate (143) to show that the difference between $q_f(L,\zeta)$ and $q_{f_L}(L,\zeta)$ is exponentially small (cf. ref. 1). Finally use the exponential convergence of $q_f(L,\zeta)$. ∎

Remark 6.1. For instance, in refs. 15 and 8 for a single-component gas and in ref. 16 for a binary gas mixture, the half-space problems are solved numerically by restricting the x_1 region (the half space) to a finite slab on the artificial boundary of which the specular reflection condition is imposed. In the above references, the influence of the truncation of region was carefully examined by changing the width of slab. The influence was found to be negligible if the width is larger than about 20 times as long as the mean free path. Proposition 6.1 and Corollary 6.1 support such a practical solution process.

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